SPRING 2023 MATH 590: EXAM 2 SOLUTIONS

Name:

Throughout V will denote a vector space over $F = \mathbb{R}$ or $\mathbb{C}$, T a linear transformation from V to V and A a matrix with entries in F.

(I) **True-False:** Write true or false next to each of the statements below. (3 points each)

(a) If A is a 3×3 real matrix whose eigenvalues are all in $\mathbb{R}$, then A is diagonalizable. False

Comment. The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has its eigenvalues in $\mathbb{R}$ but is not diagonalizable.

(b) For vectors $v_1, v_2, w \in \mathbb{R}^5$, if v_1 and v_2 are orthogonal to w, then $7v_1 + 2v_2$ is orthogonal to w. True

(c) There exists a 7×7 matrix with entries in $\mathbb{C}$ that has 81 distinct **eigenvectors**. True

Comment. Since $\mathbb{C}$ is infinite, every eigenspace (or any vector space) has infinitely many vectors.

(d) Suppose P and A are 2×2 real matrices satisfying: $P^{-1} = P^t$ and $P^{-1}AP = D$, where D is a diagonal matrix. Then A is a symmetric matrix. True

Comment. This is the converse to the Spectral Theorem for symmetric matrices over $\mathbb{R}$.

(e) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^4$, and $T(v_1) = v_1, T(v_2) = \sqrt{2} \cdot v_2, T(v_3) = e \cdot v_3$ and $T(v_4) = \pi \cdot v_4$. Then T is diagonalizable. True

Comment. Any transformation of an n diimensional space with n distinct eigenvalues is diagonalizable.

(II) **State the indicated definition, proposition or theorem.** (5 points each)

(a) State the theorem characterizing when the linear transformation $T : V \to V$ is diagonalizable.

Solution. For a linear transformation $T : V \to V$, with $\dim(V) = n$, the following are equivalent:

(i) T is diagonalizable.
(ii) $p_T(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$ for distinct $\lambda_1 \in \mathbb{R}$ and $\dim(E_{\lambda_i}) = e_i$, for $1 \leq i \leq r$.
(iii) $p_T(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$ for distinct $\lambda_i \in \mathbb{R}$ and $\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_r}) = n$.

Comment. See the Daily Update of March 22 and also Quiz 8.

(b) State the theorem giving the Gram-Schmidt process as it applies to the independent set of vectors $\{v_1, v_2, v_3\}$.

Solution. There exists an orthogonal set of vectors $\{w_1, w_2, w_3\}$ satisfying $\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{w_1, w_2, w_3\}$, where

(i) $w_1 = v_1$
(ii) $w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} \cdot w_1$.
(iii) $w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} \cdot w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} \cdot w_2$.

Comment. See the Daily Update of March 31.

(c) State the four conditions that must be satisfied by the inner product $\langle\ ,\ \rangle$ on the real vector space V.

Solution. The function from $V \times V$ to $\mathbb{R}$ taking (v, w) to $\langle v, w\rangle$ is an inner product if, it satisfies the following conditions for all $v, w \in V$ and $\lambda \in \mathbb{R}$:

(i) $\langle v, w\rangle = \langle w, v\rangle$.
(ii) $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$.
(iii) $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle = \langle v, \lambda w\rangle$.
(iv) $\langle v, v\rangle \geq 0$ and $\langle v, v, \rangle = 0$ if and only if $v = \vec{0}$.

Comment. See the Daily Update of March 29.

(III) **Short Answer**. (15 points each)

(a) Given the system of linear equations

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ -2x + 3z &= 8 \\ -3y + 9z &= 14, \end{aligned}$$

use Cramer's rule to solve for x, y, z, but do NOT calculate any of the resulting determinants.

Solution. Letting $D = \begin{vmatrix} 2 & -3 & 5 \\ -2 & 0 & 3 \\ 0 & -3 & 9 \end{vmatrix}$, we have

$$x = \frac{\begin{vmatrix} 11 & 3 & 5 \\ 8 & 0 & 3 \\ 14 & -3 & 9 \end{vmatrix}}{D} \qquad y = \frac{\begin{vmatrix} 2 & 11 & 5 \\ -2 & 8 & 3 \\ 0 & 14 & 9 \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} 2 & -3 & 11 \\ -2 & 0 & 8 \\ 0 & -3 & 14 \end{vmatrix}}{D}.$$

Comment. See the Daily Update of March 1.

(b) Given $A = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$, find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix. Be sure to explain why P is orthogonal and verify that $P^{-1}AP$ is diagonal.

Solution. We have $p_A(x) = \begin{vmatrix} x-6 & -2 \\ -2 & x-6 \end{vmatrix} = (x-6)^2 + 4 = x^2 - 12 + 32 = (x-8)(x-4)$, so that 8, 4 are the eigenvalues of A

E_8 is the nullspace of $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \overset{\text{EROs}}{\rightarrow} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, so $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for E_8.

E_4 is the nullspace of $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \overset{\text{EROs}}{\rightarrow} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, so $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a basis for E_4.

Note that $v_1 \cdot v_2 = 0$, so that v_1, v_2 are orthogonal (this always happens for a symmetric matrix with distinct eigenvalues). Thus, if $u_1 = \frac{1}{\sqrt{2}} \cdot v_1$ and $u_2 := \frac{1}{\sqrt{2}} \cdot v_2$, $\{u_1, u_2\}$ is an orthonormal basis for $\mathbb{R}^2$ and hence $P = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is an orthogonal matrix. Therefore, $P^{-1} = P^t = P$ (in this case). It follows that

$$\begin{aligned} P^{-1}AP &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 8 & 8 \\ 4 & -4 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 16 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}. \end{aligned}$$

Comment. This problem is almost exactly the same as problem (ii) given in the homework of March 3.

(c) Suppose V is the vector space of 2×2 matrices over $\mathbb{R}$ with inner product $\langle A, B \rangle := \text{trace}(A^t B)$. Find an orthonormal basis for the subspace of V spanned by $v_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $v_2 := \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

Solution. Note that $\langle v_1, v_2 \rangle = \text{trace}\{A^t B\} = \text{trace} \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} = 0$, so v_1, v_2 are orthogonal. Thus, we just have to normalize these vectors to get the required orthonormal basis. We have

$$\langle v_1, v_1 \rangle = \text{trace}\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\} = \text{trace} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

so $||v_1|| = \sqrt{2}$ and

$$\langle v_2, v_2 \rangle = \text{trace}\{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\} = \text{trace} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = 4$$

so $||v_2|| = 2$. Thus, we may take

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad u_2 = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

for the required orthonormal basis.

Comment. This problem is almost exactly the same as, but easier than, the extra homework problem given on March 31.

(IV) **Proof Problem.** Let A be a 7×7 matrix over $\mathbb{R}$ such that $P^{-1}AP = D$ for some invertible 7×7 matrix P and D the diagonal matrix with $\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2, \lambda_3, \lambda_3$ down its diagonal. Prove: The characteristic polynomial $p_A(x) = (x - \lambda_1)^2(x - \lambda_2)^3(x - \lambda_3)^2$, $\dim(E_{\lambda_1}) = 2, \dim(E_{\lambda_2}) = 3, \dim(E_{\lambda_3}) = 2$. (25 points)

Solution. For any two matrices A and B, if $P^{-1}AP = B$, for an invertible matrix P, then $p_A(x) = p_B(x)$. Thus, in our case, $p_A(x) = p_D(x) = (x - \lambda_1)^2(x - \lambda_2)^3(x - \lambda_3)^2$, since the matrix $xI_7 - D$ is a diagonal matrix. Now, the algebraic multiplicities of $\lambda_1, \lambda_2, \lambda_3$ are 2, 3, 2, respectively, and therefore, by a theorem from class, $\dim(E_{\lambda_1}) \leq 2, \dim(E_{\lambda_2}) \leq 3, \dim(E_{\lambda_3}) \leq 2$.

On the other hand, if we let $u_1 u_2, \ldots, u_7$ denote the columns of P, then $PA = [Au_1 \; Au_2 \; Au_3 \; Au_4 \; Au_5 \; Au_6 \; Au_7]$ and $AD = [\lambda_1 u_1 \; \lambda_1 u_2 \; \lambda_2 u_3 \; \lambda_2 u_4 \; \lambda_2 u_5 \; \lambda_3 u_6 \; \lambda_3 u_7]$. Since $AP = PF$, we have

$$\begin{aligned}
Au_1 &= \lambda u_1 \\
Au_2 &= \lambda_1 u_2 \\
Au_3 &= \lambda_2 u_3 \\
Au_4 &= \lambda_2 u_4 \\
Au_5 &= \lambda_2 u_5 \\
Au_6 &= \lambda_3 u_6 \\
Au_7 &= \lambda_3 u_7
\end{aligned}$$

Since $u_1, \ldots, u_7$ are the columns of an invertible matrix, they are linearly independent. Thus u_1, u_2 are linearly independent in E_{λ_1}, and hence $\dim(E_{\lambda_1}) \geq 2$; u_3, u_4, u_5 are linearly linearly independent in E_{λ_2}, and hence $\dim(E_{\lambda_2}) \geq 3$; u_6, u_7 are linearly independent in E_{λ_3}, and thus $\dim(E_{\lambda_3}) \geq 2$.

It now follows that $\dim(E_{\lambda_1}) = 2, \dim(E_{\lambda_2}) = 3, \dim(E_{\lambda_3}) = 2$.

Bonus Problem. For ten bonus points, prove one (**and only one**) of the following bonus problems. In order to receive any bonus points, your answer must be completely (or, very close to completely) correct.

1. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a symmetric linear transformation. Prove that $[T]_\alpha^\alpha$ is a symmetric matrix, for every orthonormal basis $\alpha \subseteq \mathbb{R}^2$. Give an example where this fails, if α is not an orthonormal basis.

Solution. Suppose $\alpha = \{u_1, u_2\}$ is an orthonormal basis for $\mathbb{R}^2$ and $T(u_1) = au_1 + bu_2$, $T(u_2) = cu_1 + du_2$. It follows that $[T]_\alpha^\alpha = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

We also have

$$T(u_1) \cdot u_2 = (au_1 + bu_2) \cdot u_2 = a(u_1 \cdot u_2) + b(u_2 \cdot u_2) = a0 + b1 = b,$$

and moreover,

$$u_1 \cdot T(u_2) = u_1 \cdot (cu_1 + du_2) = c(u_1 \cdot u_1) + d(u_1 \cdot u_2) = c1 + d0 = c.$$

Since $T(u_1) \cdot u_2 = u_1 \cdot T(u_2)$, it follows that $b = s$, showing that $[T]_\alpha^\alpha$ is symmetric.

Now consider $T(x, y) = (x + 2y, 2x + y)$, a symmetric linear transformation. If we let $v_1 = (1, 1)$ and $v_1 = (1, 0)$, then $\beta = \{v_1, v_2\}$ is a basis for $\mathbb{R}^2$ (since the corresponding determinant is not zero). On the other hand, $T(v_1) = (3, 3) = 3 \cdot v_1 + 0 \cdot v_2$ and $T(v_2) = (1, 2) = 2 \cdot v_1 - 1 \cdot v_2$, so that $[T]_\beta^\beta = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$, which is not a symmetric matrix.

2. Suppose $\lambda_1, \ldots, \lambda_r \in F$ are distinct and $T(v_i) = \lambda_i v_i$, for non-zero vectors $v_1, \ldots, v_r$. Then $v_1, \ldots, v_r$ are linearly independent.

Solution. Suppose by way of contradiction, $v_1, \ldots, v_r$ are **not** linearly independent. Then there is a linear combination of these vectors with at least one coefficient not zero. Among all such combinations, take one with the fewest number of elements. After re-indexing, we may assume we have a linear dependence relation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_s v_s = \vec{0}$$

of shortest possible length. Thus, each $\alpha_i \neq 0$.

If we apply $(T - \lambda_1)$ to the equation above, and use the fact that $T - \lambda_1 I$ is a linear transformation, we have

$$\begin{aligned}
\alpha_1(T - \lambda_1 I)(v_1) + \alpha_2(T - \lambda_1 I)(v_2) + \cdots + \alpha_s(T - \lambda_1 I)(v_s) &= \vec{0} \\
\alpha_1(T(v_1) - \lambda_1 v_1) + \alpha_2(T(v_2) - \lambda_1 v_2) + \cdots + \alpha_s(T(v_s) - \lambda_1 v_s) &= \vec{0} \\
\alpha_1(\lambda_1 v_1 - \lambda_1 v_1) + \alpha_2(\lambda_2 v_2 - \lambda_1 v_2) + \cdots + \alpha_s(\lambda_s v_s - \lambda_1 v_s) &= \vec{0} \\
\alpha_2(\lambda_2 - \lambda_1)v_2 + \cdots + \alpha_s(\lambda_s - \lambda_1)v_s &= \vec{0}.
\end{aligned}$$

Since each $\alpha_i(\lambda_i - \lambda_1) \neq 0$, the last equation above gives a dependence relation of length less than s. This contradiction implies that there can be no dependence relation among $v_1, \ldots, v_r$, i.e., $v_1, \ldots, v_r$ are linearly independent.